Adams-Sears iterative procedure, which is straightforward once the solutions (14) and (15) are known (e.g., see Ref. 2 and Ref. 4, pp. 34-40)

We remark that the solutions (14) and (15) appear to be new in the literature, which can be otherwise obtained from more complicated methods such as integral transform. Furthermore, we note that these solutions are strictly near-field solutions, derived for bodies/wings of slender ratio upon a very small departure from the nonlifting flow (hence, a thickness dominated problem). They should be therefore distinguished from the recent work of Cheng and Hafez[11] in which three-dimensionality is considered for the nonlinear transonic flowfield involving different degrees of lift control.

Conclusions

A much simpler derivation for the not-so-slender wing/body potential has been presented as a result of the uncovered asymptotic property of the line source structure near the body axis. Based on the asymptotic equations formulation, a general recurrence formula is obtained which can yield readily the NSSB potentials up to arbitrary order in r. Formulas Eq. (8a) and Eq. (11) are generally valid for a smooth body in all regimes of the potential flow insofar as the governing equations are linear. Also, the linear operator Λ is restricted to performing differentiation only in the x-direction.

The present approach can be readily extended to the unsteady flow analysis. To obtain the oscillatory NSSB potential, however, the spatial influence function, $g(x)$ [see Eq. (3a)], corresponding to the axisymmetric pulsating solutions, must be provided by other means of derivation. In fact, these pulsating solutions have been previously established by Platzer[12] for subsonic and supersonic cases and generalized by Liu, et al.,[6] [Eqs. (2.17)-(2.20), p. 3] to include the linearized transonic flow case. It is demonstrated in Ref. 13 that following the present approach the oscillatory NSSB potential for the linearized transonic flow can be derived within a few steps. Further work concerning the unsteady-flow extension will be reported in a forthcoming Note.

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References

[1]Ward, G.N., *"Linearized Theory of High Speed Flow,"* Cambridge University Press, Oxford, England, 1955, pp. 165-186.

[2]Adams, M.C. and Sears, W.R., "Slender Body Theory-Review and Extension," *Journal of Aeronautical Sciences*, Vol. 20, No. 2, Feb. 1953, pp. 85-98.

[3]Miles, J.W., *"The Potential Theory of Unsteady Supersonic Flow,"* Cambridge University Press, Oxford, England 1959, pp. 23, 36, and 62.

[4]Landahl, M.T., *"Unsteady Transonic Flow,"* Pergamon Press, New York, 1961, pp. 24 and 32.

[5]Platzer, M.F. and Hoffman, G.H., "Quasi-Slender Body Theory for Slowly Oscillating Bodies of Revolution in Supersonic Flow," NASA TND-3440, 1966.

[6]Liu, D.D., Platzer, M.F., and Ruo, S.Y., "On the Calculation of Static and Dynamic Stability Derivatives for Bodies of Revolution at Subsonic and Transonic Speed," AIAA Paper 70-190, New York City, Jan. 1970.

[7]Oswatitsch, K. and Keune, F., "Flow Around Bodies of Revolution at Mach Number One," *Proceedings of the Conference on High-Speed Aeronautics,* Polytechnic Institute of Brooklyn, New York, 1955, pp. 113-131.

[8]Cole, J.D. and Royce, W.W., "An Approximate Theory in the Pressure Distribution and Wave Drag of Bodies of Revolution at Mach Number One," Sixth Annual Conference on Fluid Dynamics, University of Texas, 1959, pp. 254-276.

[9]Burg, K., "Eine neue Methode Zur Berechnung der Schallnahen Stromung un den Flugel Kleiner Spannweite," *ZAMM* Vol. 50, 1970, pp. T170-T173.

[10]Kimble, K.R., Liu, D.D., Ruo, S.Y., and Wu, J.M., "Unsteady Transonic Flow Analysis for Low Aspect Ratio, Pointed Wings," *AIAA Journal,* Vol. 12, April 1974, pp. 516-522.

[11]Cheng, H.K. and Hafez, M.M., "Transonic Equivalence Rule: A Nonlinear Problem Involving Lift," *Journal of Fluid Mech.,* Vol. 72, Pt. 1, 1975, pp. 161-187.

[12]Platzer, M., "Nicht angestellte pulsierende Korper kleiner Streckung in Unter- and Uberschallstromung," *ACTA Mechanica* Vol. II/1, 1966, pp. 70-89.

[13]Liu, D.D., Platzer, M.F., and Ruo, S.Y., *"Unsteady Linearized Transonic Flow Analysis for Slender Bodies,"* submitted to AIAA Journal.

[14]Tsien, H.S., "Supersonic Flow Over an Inclined Body of Revolution," *Journal of Aeronautical Sciences* 1938, pp. 480-483.

Efficiency of Navier-Stokes Solvers

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THIS paper describes numerical experiments used to evaluate the relative efficiency of some finite-difference methods for the solution of the vorticity-stream function form of the two-dimensional incompressible Navier-Stokes equations. The comparisons were drawn by recording the CPU time required to obtain a solution as well as the accuracy of this solution using five numerical methods: central differences, first-order upwind differences, second-order upwind differences, exponential differences (these four methods incorporate the Gauss-Seidel iterative procedure), and an ADI solution of the central difference equations. The mesh sizes used were 11×11, 21×21, and 31×31.

Solutions were obtained for two test cases: a recirculating eddy inside a square cavity with a moving top, and an impinging jet flow. Solutions for various Reynolds numbers were obtained, with emphasis on high Reynolds number flows.

Equations and Test Problems

The vorticity-stream function formulation of the two-dimensional Navier-Stokes equations is given by the vorticity equation

$$\frac{\partial}{\partial x}(u\zeta) + \frac{\partial}{\partial y}(v\zeta) = \nu\left[\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right] \tag{1}$$

and the stream function equation

$$-\zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \tag{2}$$

where ζ is the vorticity, ψ is the stream function, u, v, x, y are the velocity components and spatial coordinates, respectively, and ν is the kinematic viscosity. The velocity components are

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calculated by

$$u = \frac{\partial \psi}{\partial y}\ ;\ v = -\frac{\partial x}{\partial x} \tag{3}$$

Two test cases were treated. In the first case the recirculating flow inside a square cavity with a moving top was calculated. The boundary conditions for this problem are that the normal velocity vanishes on all four walls, whereas, the tangential velocity is zero on three stationary walls and is unity on the moving wall. This problem has been chosen because of the great number of solutions available for it, which enable direct comparison of new solutions with previous ones.[1,2]

The second problem was that of a plane jet impinging normally onto a flat infinite surface. This is a symmetric problem, and only half of the jet ought to be considered. Boundary conditions for this problem are: zero tangential and normal velocity on the surface, zero vorticity and stream function on the symmetry axis, prescribed vorticity and stream function profiles on the upstream boundary, and zero vorticity and stream function gradients normal to the downstream boundary. For further details the reader is referred to Ref. 1.

Finite-Difference Equations

The differential equations were approximated by finite differences on a uniform mesh. The usual five-point scheme was used for the Laplacian, and central differences were used for the velocities in Eq. (3). Four schemes were used for the vorticity transport terms: a conservative centered scheme,[2] a conservative "upwind" scheme,[3] an exponential difference scheme,[4,6] and a nonconservative second-order "upwind" scheme analyzed for a one-dimensional case by Price et al.[5] The fourth scheme which is not well known is rederived here for convenience in a simple way. Consider the derivative $\partial\zeta/\partial x$ at a net point (x, y) when the velocity u is positive. Expansion about $x-h/2$ gives

$$\frac{\partial \zeta}{\partial x}(x,y) = \frac{\partial \zeta}{\partial x}\left(x-\frac{h}{2},y\right) + \frac{h}{2}\,\frac{\partial^2 \zeta}{\partial x^2}\left(x-\frac{h}{2},y\right) + O(h^2) \tag{4}$$

but

$$h\frac{\partial^2 \zeta}{\partial x^2}\left(x-\frac{h}{2},y\right) = h\frac{\partial^2 \zeta}{\partial x^2}(x-h,y) + O(h^2)$$

and using only "upwind" values, it follows that

$$\frac{\partial \zeta}{\partial x}(x,y) = \frac{\zeta(x,y)-\zeta(x-h,y)}{h} + \frac{\zeta(x,y)-2\zeta(x-h,y)+\zeta(x-2h,y)}{2h} + O(h^2) \tag{5}$$

which is equivalent to

$$\frac{\partial \zeta}{\partial x}(x,y) = 2\frac{\partial \zeta}{\partial x}\left(x-\frac{h}{2},y\right) - \frac{\partial \zeta}{\partial x}(x-h,y) + O(h^2) \tag{6}$$

Implementation of Eq. (6) is particularly simple in an iterative scheme where required values are obtained from previous iterations. Thus, when the iteration converges second-order accuracy is retained.

Special treatment is required when $(x-h,\ y)$ falls on a boundary. On an inflow boundary the normal derivative of the vorticity is known, on an impermeable boundary the normal velocity vanishes and $u(x, y)$, at the first mesh point, is $O(h)$ and therefore the simple "upwind"scheme can be used still keeping the error $O(h^2)$.

A linearized stability analysis for the fourth scheme, indicates unconditional stability for a Gauss-Seidel iteration for all Reynolds numbers.

To summarize this section the coefficients of the finite-difference equation for each scheme will be summarized. The coefficients of the upwind schemes are, of course, dependent on the signs of the velocity components u and v. Therefore, to simplify the presentation, only the coefficients for positive u and v are given here. The coefficients for other cases may easily be derived. All of the schemes yield equations of the form

$$C_P\zeta_P = C_N\zeta_N + C_S\zeta_S + C_E\zeta_E + C_W\zeta_W + R_P \tag{7}$$

The C coefficients are given in Table 1, together with the maximal Reynolds number allowing a stable solution of Eq. (7) by the Gauss-Seidel method.

A second-order accurate finite-difference equation for the solid wall vorticity was used. For an impermeable wall which moves in its plane at a velocity u_W, the equation is

$$\zeta_p = -\left(\frac{3}{h^2}\right)(\psi_N - \psi_P + hu_W) - \frac{1}{2}\zeta_N + O(h^2) \tag{8}$$

Table 1 Coefficients of the finite-difference vorticity equation[a]

	Central	Upwind, first order $u>0; v>0$	Upwind, second order $u>0; v>0$	Exponential
C_N	$1-R_N/2$	1	1	E_yD
C_S	$1+R_S/2$	$1+R_S$	$1+2Rs$	$(1-E_y)D$
C_E	$1-R_E/2$	1	1	E_x
C_W	$1+R_W/2$	$1+R_W$	$1+2R_W$	$1-E_x$
C_P	4	$4+R_x+R_y$	$4+2R_x+2R_y$	$1+D$
R_P	0	0	$\frac{h^2}{\nu}\left[\left(\frac{\partial\zeta}{\partial x}\right)_W+\left(\frac{\partial\zeta}{\partial y}\right)_S\right]$	0
Stability condition	All R's smaller then 2	Unconditional	Unconditional	Unconditional
Accuracy	h^2	h	h^2	h^2
Conservation	Yes	Yes	No	No

[a] $R_N = V_Nh/\nu$; $R_S = V_Sh/\nu$; $R_E = u_Eh/\nu$; $R_W = u_Wh/\nu$; $R_X = u_Ph/\nu$; $R_Y = v_Ph/\nu$; $E_x = [\exp(R_X) - 1]/[\exp(2R_X) - 1]$; $E_y = [\exp(R_y) - 1]/[\exp(2R_y) - 1]$; $D = (v/u)[(1-2E_x)/(1-2E_y)]$.

where P is a point on the wall, and N is the node adjacent to P inside the field. An estimate of the accurate solution was obtained by applying Richardson's extrapolation to the central difference results.

Results

Both the square cavity and impinging jet flow have already been reported in the literature and need not be described in detail here. Instead attention will be given to the comparison between the different methods. As both flows are complex, a detailed comparison of various solutions would be very lengthy. Therefore, only typical quantitites for the flow in question will be considered. For the cavity flow a representative quantity is the maximum stream function in the field, which is a measure of the strength of the captive eddy inside the cavity. The computed value of this quantity is given in Table 2 for different methods and for different Reynolds numbers. It has been shown in previous investigations that the maximum value of the stream function is about 0.1. In Table 2 it can be seen that central difference solutions using ADI method and 31×31 mesh are fairly accurate. The second-order upwing method shows a deteriorated accuracy at high Reynolds numbers, although the 21×21 mesh produces much better accuracy than the 11×11 mesh. The first order upwind differences are significantly worse than other results.

The results in Table 2 allow comparison between different methods on the basis of the price required to obtain a given accuracy. The results show that second-order central difference is the cheapest method at the given Reynolds numbers. However, ADI is only slightly more expensive for the same accuracy. The second-order upwind method is the most expensive of all second-order methods, but even so it is not much more expensive than the three above-mentioned second-order methods at low errors. At the low accuracy

Table 2 Error of maximal stream function in the square cavity (compared with results of a Richardson extrapolation)

Mesh size									Method
31 × 31			21 × 21			11 × 11			
Error of max stream function	CPU sec	No. of iterations	Error of max stream function	CPU sec	No. of iterations	Error of max stream function	CPU sec	No. of iterations	
				Re = 1					
0.47	117.7	555	0.96	21.64	236	3.3	1.74	72	ADICC[a]
0.77	95.4	684	0.96	19.90	306	3.3	1.42	76	CC[b]
0.47	107.3	682	0.96	23.85	305	3.3	1.81	76	EXP[c]
1.43	105.3	686	1.9	20.85	308	4.2	1.46	75	UFO[d]
0.47	122.8	688	0.97	24.64	310	3.4	1.53	77	USO[e]
				Re = 100					
0.85	100.8	480	2.3	25.5	274	8.1	1.85	76	ADICC
0.85	82.9	592	2.3	15.3	237	8.1	1.17	64	CC
2.49	124.3	584	6.1	24.6	244	19.7	1.76	62	EXP
1.38	106.5	700	2.6	17.0	248	10.1	1.32	66	UFO
1.85	102.3	579	3.9	24.0	303	9.4	1.85	92	USO
				Re = 1000					
6.4	183.2	905	14.4	34.2	376	30.7	2.92	120	ADICC
29.0	207.4	1330	33.2	49.7	660	49.5	2.41	129	UFO
8.8			16.3			42.2			USO

[a] ADICC – same as CC but with ADI. [b] CC – central differences. [c] EXP – exponential differences. [d] UFO – upwind differences, first order, conservative. [e] USO – upwind differences, second order, nonconservative.

Table 3 Error of the maximal wall and exit vorticity in the impinging jet (compared with results of Richardson extrapolation)

Mesh size												Method[a]
31 × 31				21 × 21				11 × 11				
error of max exit vorticity	error of max wall vorticity	CPU sec	No. of iterations	error of max exit vorticity	error of max wall vorticity	CPU sec	No. of iterations	error of max exit vorticity	error of max wall vorticity	CPU sec	No. of iterations	
						Re = 100						
1.29	0.91	105.6	480	2.91	2.04	19.01	203	11.8	6.9	1.5	55	ADICC
1.23	0.55	173.4	780	3.7	1.36	34.5	362	15.5	5.25	2.72	106	EXP
3.70	3.0	126.6	822	7.0	3.95	27.1	392	19.6	6.7	2.16	113	UFO
0.32	1.33	157.0	857	1.29	1.39	35.6	465	13.4	1.7			USO
						Re = 1000						
7.3	5.6	103.0	472	17.7	12.5	22.9	246	45.8	35.4	4.64	185	ADICC
16.3	9.7	99.0	439	27.3	22.6	20.6	211	58.9	61.3	1.81	67	EXP
19.8	16.9	78.9	502	34.8	25.0	17.9	252	60.9	53.6	1.70	85	UFO
4.7	2.7	120.6	653	12.1	14.1			54.0	51.5			USO

[a] Abbreviations as in Table 2.

corresponding to 11×11 mesh the second-order upwind scheme requires about the same computer time as the other methods, but its error is much greater. The first-order upwind scheme seems to have larger errors at all meshes.

Altogether it may be concluded that whenever the central difference scheme is stable it requires less computer time to yield a given accuracy. The stability of central differences is increased by the ADI method but at an increased CPU time. The second-order upwind scheme is stable even at $Re = 5000$ (compared with a limit of 1000 for the central differences). It is always somewhat less efficient than central differences. Compared with the first-order upwind scheme it is slightly less efficient at low Reynolds numbers and much more efficient at high Reynolds numbers.

For the impinging jet the typical representative quantity chosen was the maximum vorticity at the downstream boundary of the control volume. This quantity is meaningful only at high Reynolds number, where it should be somewhat lower than the maximum vorticity on the upstream boundary of the control volume. This upper bound was 4.59 in the present calculations.

As in the cavity flow, central differences are most accurate, but in this case the second-order upwind scheme seems to be nearly as good as central differences, and considerably better than the first-order upwind and the exponential schemes, with respect to its accuracy and CPU time demands.

Conclusions

The main conclusion of this study is that the second-order upwind scheme appears to have the potential of yielding sufficient accuracy at an acceptable price (in spite of the fact that it is nonconservative). However, it may become necessary to improve its stability by the application of ADI techniques.

As might have been expected, central difference methods offer the cheapest way to obtain a given accuracy whenever they are stable. The first-order upwind scheme was not usually as accurate as the second-order schemes, but its accuracy was not as bad as might have been expected on the basis of accuracy analysis. There seems to be little point in using the exponential scheme.

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References

[1]Runchal, A. K., Spalding, D. B., and Wolfshtein, M., "Numerical Solutions of the Elliptic Equations for Transport of Vorticy, Heat and Matter in Two-Dimensional Flow," *The Physics of Fluids,* Supplement II, 1969, pp. II/21—II/28.

[2]Torrance, K., Davis, R., Eike, K., Gill, P., Gutman, D., Hsui, A., Lyons, S., and Zien, H., "Cavity Flows Driven by Buoyancy and Shear," *Journal of Fluid Mechanics,* Vol. 51, 1972, pp. 221—331.

[3]Gosman, A. D., Pun, W. M., Runchal, A. K., Spalding, D. B., and Wolfshten, M., *Heat and Mass Transfer in Recirculating Flows,* Academic Press, London, 1969.

[4]Dennis, S. C. R., "The Numerical Solution of the Vorticity Transport Equation," *Proceedings of 3rd International Conference on Numerical Methods in Fluid Dynamics,* Vol. 2, 1972, Paris, pp. 120—129.

[5]Price, H. S., Varga, R. S., and Warren, J. E., "Applications of Oscillation Matrices to Diffusion-Correction Equations," *Journal of Mathematics and Physics,* Vol. 45, 1966, pp. 301—311.

[6]Raithby, G. D. and Torrance, K. W., "Upstream-Weighted Differencing Schemes as on Application to Elliptic Problems Involving Fluid Flow," *Computers and Fluids,* Vol. 2, 1974, pp. 191—206.

Experimental Evaluation of Sampling Bias in Individual Realization Laser Anemometry

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Nomenclature

N = number of velocity realizations
P = pressure
Re = Reynolds number based on mass average velocity and hydraulic diameter
T = integration time
$\bar{T}_D$ = average Doppler period for N realizations
t = time
U = streamwise velocity
$\bar{U}$ = time-average fluid velocity
U_C = period average velocity
U_B = frequency average velocity
U_i = an individual velocity realization
U_τ = $(\tau_w/\rho)^{1/2}$ = shear velocity
U^+ = average streamwise velocity nondimensionalized with U_τ
y = distance of the center of the probe volume from the wall
y^+ = yU_τ/v
y_0 = correction to the wall location
b_p = slope of the average velocity profile deduced from pressure gradient measurements
Δ = change in a quantity
$\theta/2$ = beam intersection half-angle
λ = wavelength of laser light
μ = absolute viscosity
v = kinematic viscosity
ρ = density
τ_w = wall shear stress deduced from pressure gradient measurements
ω_i = weighting function

Introduction

ONE of the principal quantities of interest in turbulent flow measurements is the time-average velocity at a point

$$\bar{U} = \frac{1}{T}\int_t^{t+T} U(t)\,\mathrm{d}t \tag{1}$$

This is a straightforward quantity to evaluate experimentally when there is a temporally continuous record of the velocity. However, when the record is discontinuous, as it is when the concentration of scattering particles is dilute and a laser Doppler anemometer is operated in the individual realization or counting mode,[1,2] a precise estimate of the time-average velocity becomes difficult. For an unbiased histogram of random, independent velocity realizations the time-average velocity can be estimated from the ensemble average of the velocity realizations by

$$U_B = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{2}$$

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